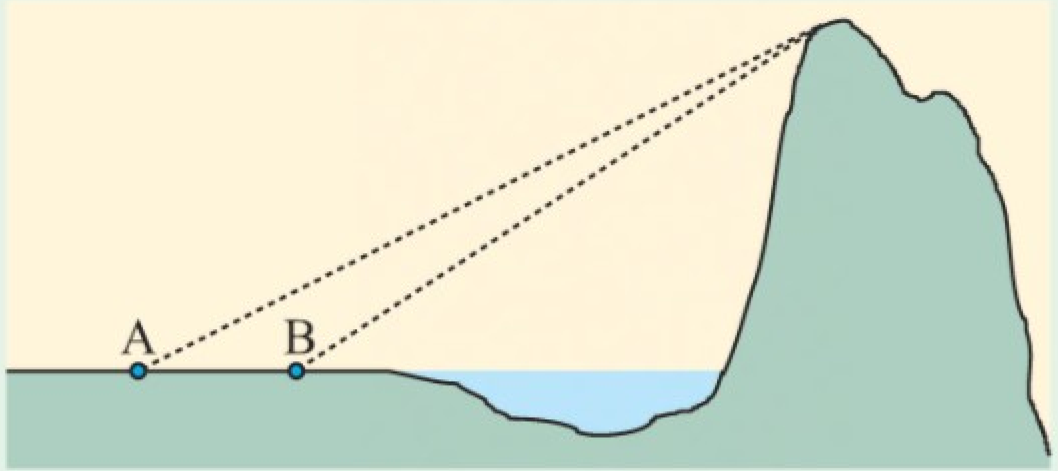
Trigonometry

OPENING PROBLEM

A surveyor is standing on horizontal ground, and wishes to find the height of a mountain on the other side of a lake. He uses a theodolite to accurately measure:



- the angle of elevation from the horizontal ground at A up to the top of the mountain, to be 33.7°
- the angle of elevation from the horizontal ground at B up to the top of the mountain, to be 41.6°
- the distance from A to B to be 400 m.

Things to think about:

a Can you draw a labelled diagram of the situation showing all information given?
b Can you solve this problem using right angled triangle trigonometry?
c Can you solve this problem faster using the sine or cosine rules?

Trigonometry is a branch of mathematics that deals with the relationship between the side lengths and angles of triangles.

We can apply trigonometry in engineering, astronomy, architecture, navigation, surveying, the building industry, and in many other branches of applied science.

HISTORICAL NOTE

ASTRONOMY AND TRIGONOMETRY

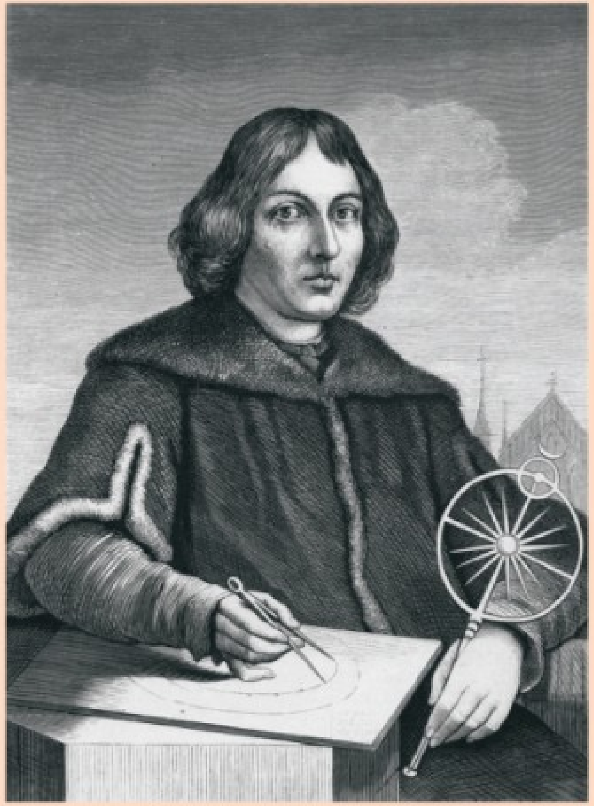
The Greek astronomer **Hipparchus** (140 BC) is credited with being the founder of trigonometry. To aid his astronomical calculations, he produced a table of numbers in which the lengths of chords of a circle were related to the length of the radius.

Ptolemy, another great Greek astronomer of the time, extended this table in his major published work *Almagest*, which was used by astronomers for the next 1000 years. In fact, much of Hipparchus' work is known through the writings of Ptolemy. These writings found their way to Hindu and Arab scholars.

Aryabhata, a Hindu mathematician in the 5th and 6th Century AD, constructed a table of the lengths of half-chords of a circle with radius one unit. This was the first table of **sine** values.

In the late 16th century, **Georg Joachim de Porris**, also known as **Rheticus**, produced comprehensive and remarkably accurate tables of all six trigonometric ratios, three of which you will learn about in this chapter. These involved a tremendous number of tedious calculations, all without the aid of calculators or computers.

Rheticus was the only student of **Nicolaus Copernicus**, and helped his tutor publish his work *De revolutionibus orbium coelestium* (On the Revolutions of the Heavenly Spheres).



Nicolaus Copernicus

A TRIGONOMETRIC RATIOS

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In previous years we have defined the following basic trigonometric ratios for right-angled triangles:

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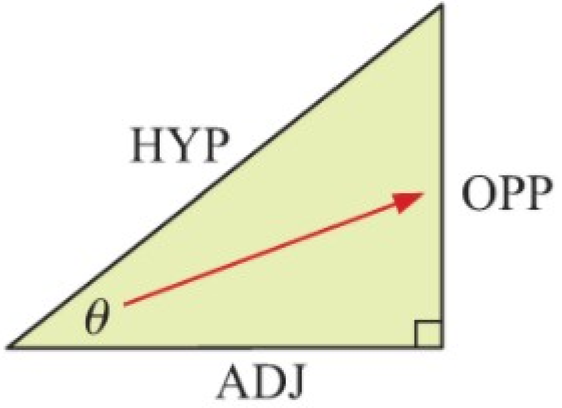
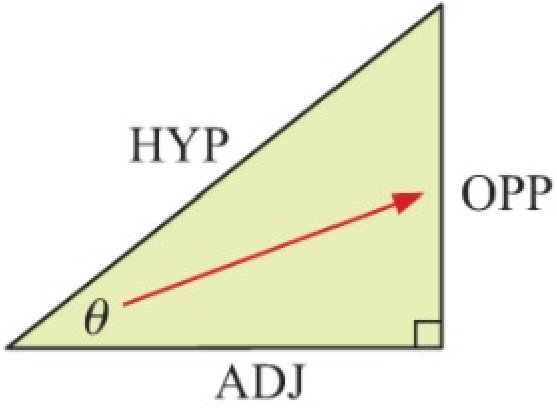


$$\sin\theta = \frac{\textbf{OPP}}{\textbf{HYP}}, \quad \cos\theta = \frac{\textbf{ADJ}}{\textbf{HYP}}, \quad \tan\theta = \frac{\textbf{OPP}}{\textbf{ADJ}}$$

A TRIGONOMETRIC RATIOS

In previous years we have defined the following basic trigonometric ratios for right-angled triangles:



$$\sin\theta = \frac{\text{OPP}}{\text{HYP}}, \quad \cos\theta = \frac{\text{ADJ}}{\text{HYP}}, \quad \tan\theta = \frac{\text{OPP}}{\text{ADJ}}$$

We can use these ratios to find unknown side lengths and angles in right angled triangles.

FINDING SIDE LENGTHS

Suppose we are given the angles of a right angled triangle, and the length of a side. We can use the trigonometric ratios to find the other side lengths.

Step 1: Redraw the figure and mark on it HYP, OPP, and ADJ relative to a given angle.

Step 2: Choose an appropriate trigonometric ratio, and construct an equation.

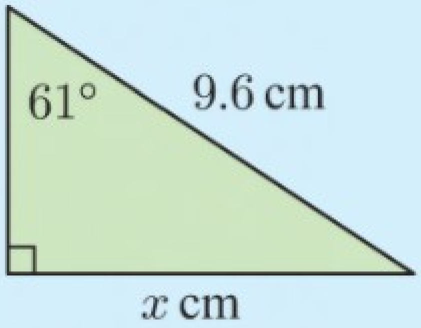
Step 3: Solve the equation to find the unknown side length.

Example 1

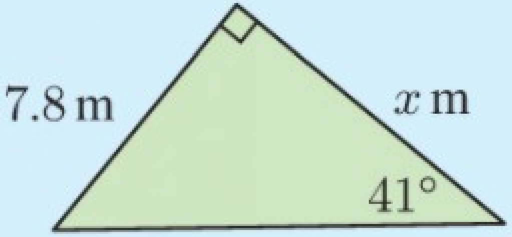
Self Tutor

Find x, rounded to 3 significant figures:

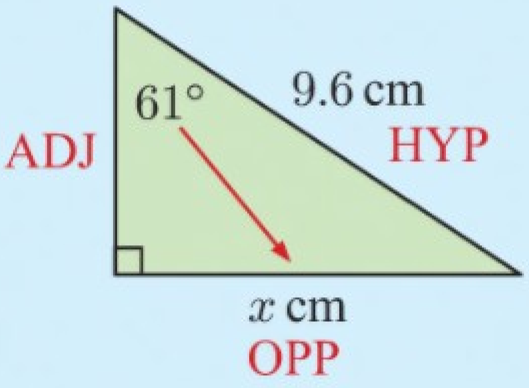
a



b



a The relevant sides are OPP and HYP, so we use the *sine* ratio.

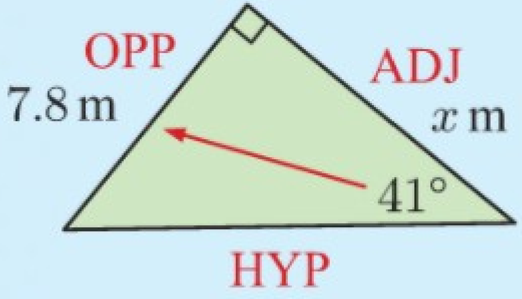


$$\sin 61^\circ = \frac{x}{9.6} \qquad \{\sin\theta = \frac{\text{OPP}}{\text{HYP}}\}$$

$$\therefore \sin 61^\circ \times 9.6 = x \qquad \{\text{multiplying both sides by } 9.6\}$$

$$\therefore x \approx 8.40 \qquad \{\text{calculator}\}$$

b The relevant sides are OPP and ADJ, so we use the *tangent* ratio.



$$\tan 41^\circ = \frac{7.8}{x} \qquad \{\tan\theta = \frac{\text{OPP}}{\text{ADJ}}\}$$

$$\therefore x \times \tan 41^\circ = 7.8 \qquad \{\text{multiplying both sides by } x\}$$

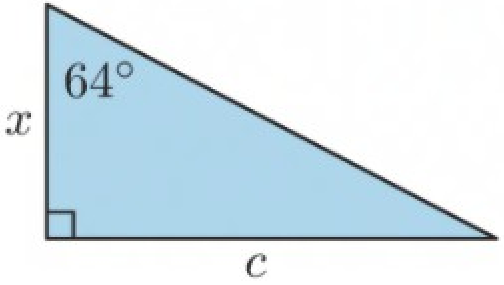
$$\therefore x = \frac{7.8}{\tan 41^\circ} \qquad \{\text{dividing both sides by } \tan 41^\circ\}$$

$$\therefore x \approx 8.97 \qquad \{\text{calculator}\}$$

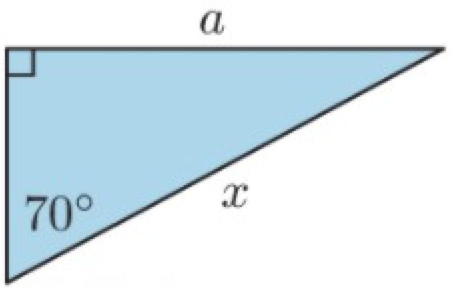
EXERCISE 12A.1

1 Construct a trigonometric equation connecting the angle with the sides given:

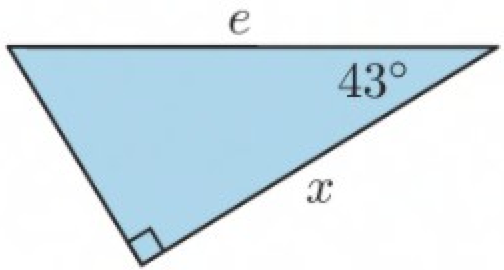
a



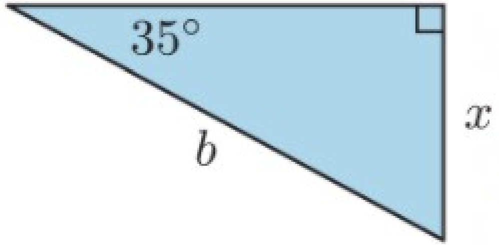
b



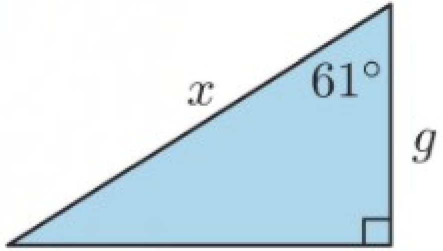
c



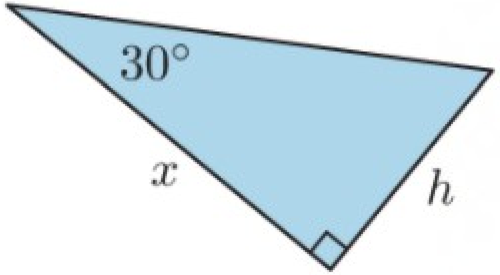
d



e

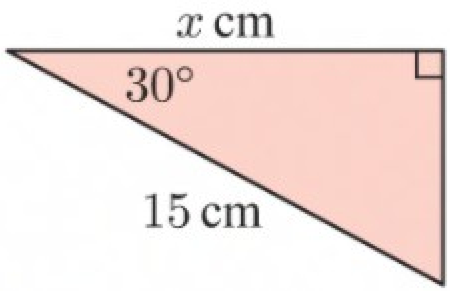


f

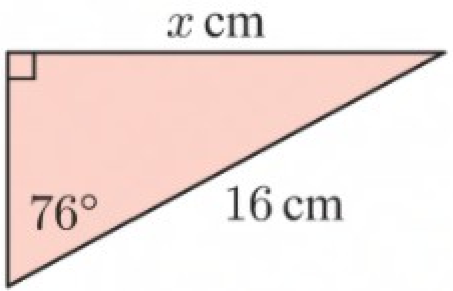


2 Find x, rounded to 2 decimal places:

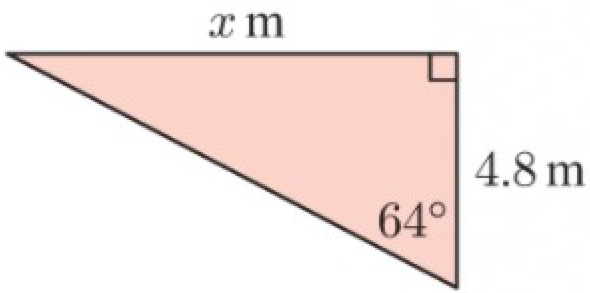
a



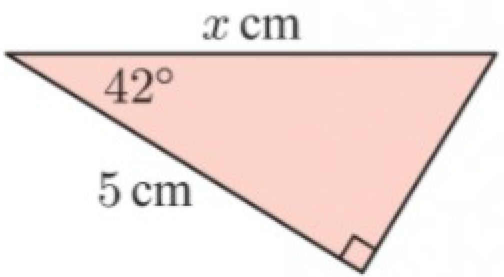
b



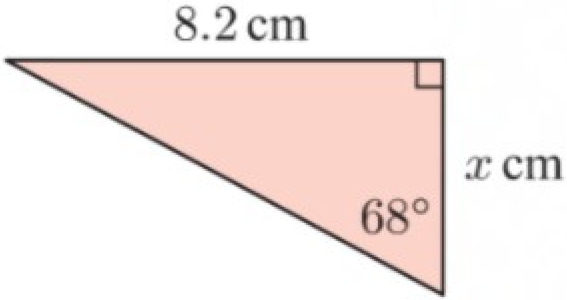
c



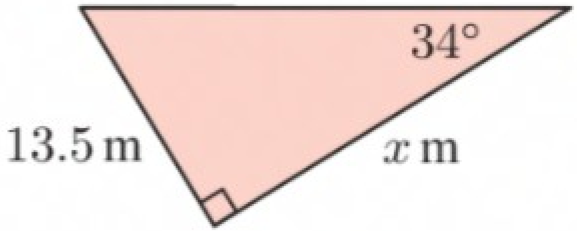
d


x cm
42°
5 cm

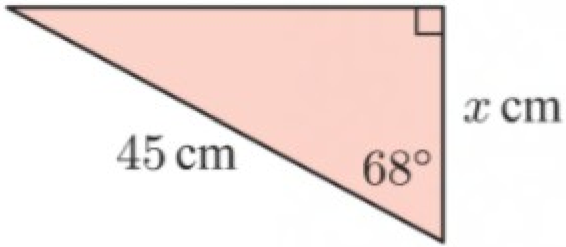
e


8.2 cm
x cm
68°

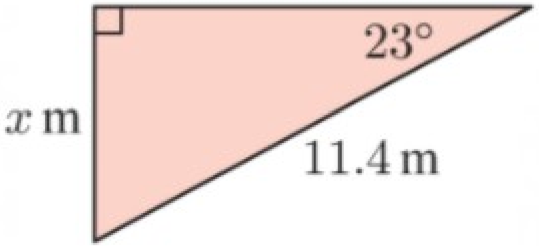
f


34°
13.5 m
x m

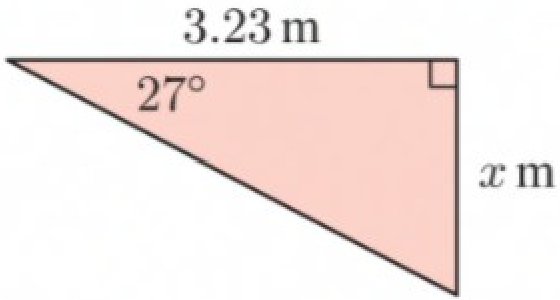
g


x cm
45 cm
68°

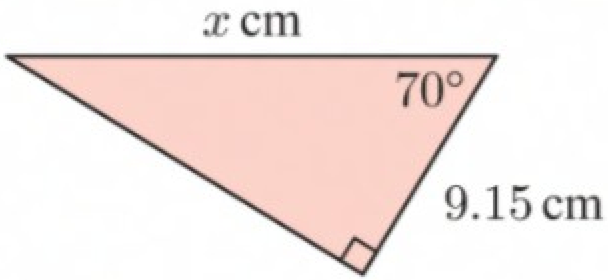
h


23°
x m
11.4 m

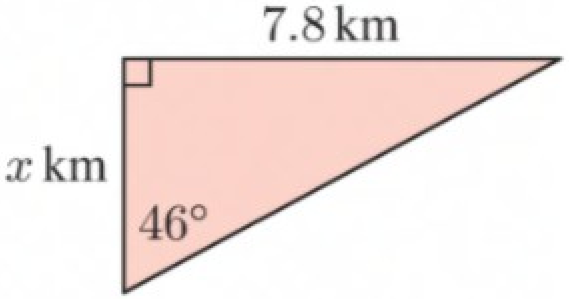
i


3.23 m
27°
x m

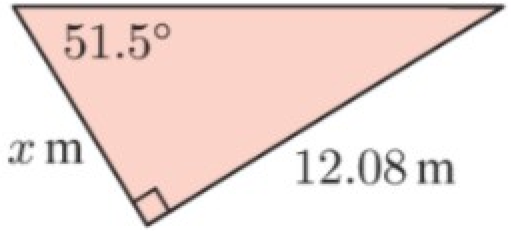
j


x cm
70°
9.15 cm

k

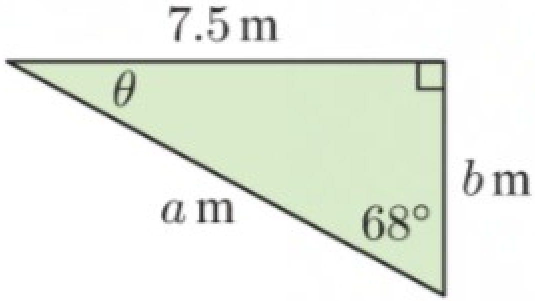

7.8 km
x km
46°

l

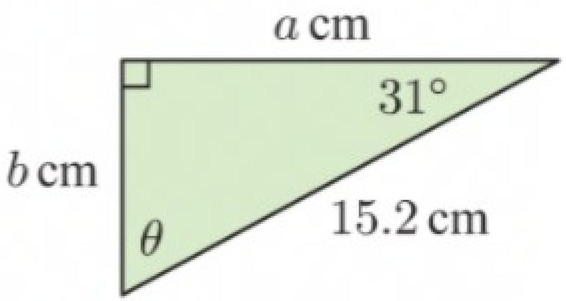

51.5°
x m
12.08 m

3 Find *all* the unknown angles and sides of:

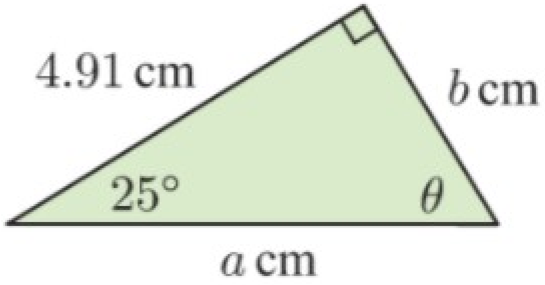
a



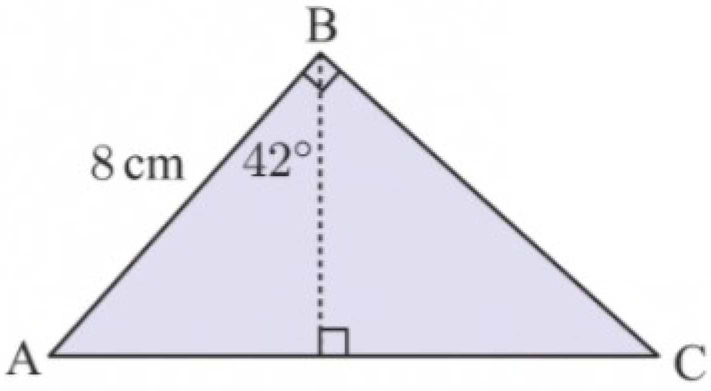
b



c



4 Find the perimeter of triangle ABC.

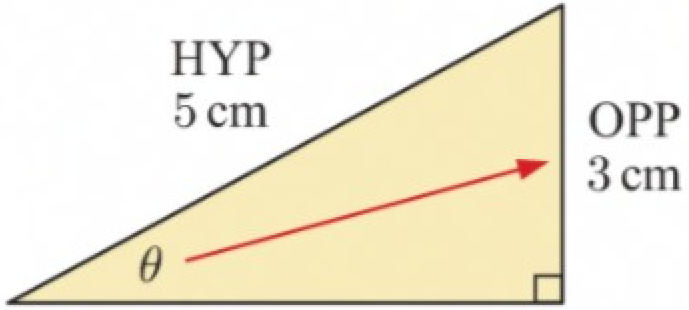


FINDING ANGLES

If we know two side lengths of a right angled triangle, we can use trigonometry to find the angles.

In the right angled triangle shown, $\sin\theta = \frac{3}{5}$.

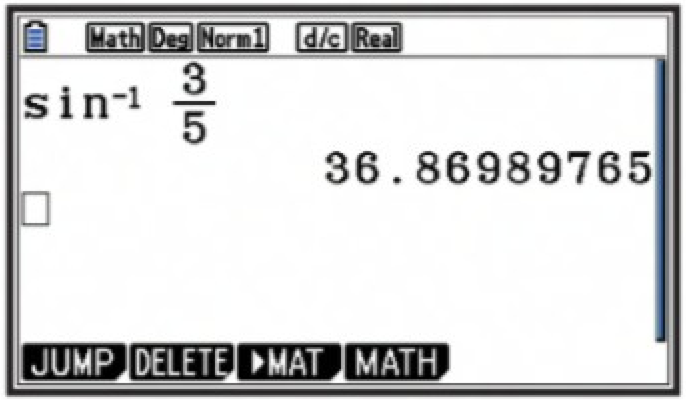
We say that θ is the **inverse sine** of $\frac{3}{5}$, and write $\theta = \sin^{-1}(\frac{3}{5})$.



We can use a calculator to evaluate inverse sines. Click on the icon for instructions.

For the right angled triangle above, we find $\theta \approx 36.9°$.

GRAPHICS
CALCULATOR
INSTRUCTIONS



$\sin^{-1}\frac{3}{5}$

36.86989765

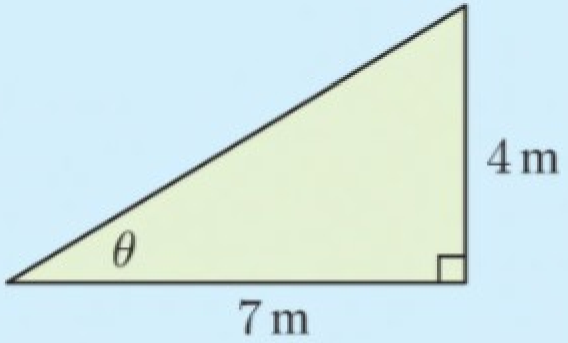
We define **inverse cosine** and **inverse tangent** in a similar way.

Example 2

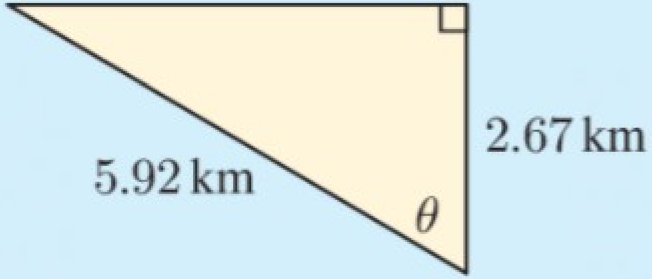
Self Tutor

Find the measure of the angle marked θ:

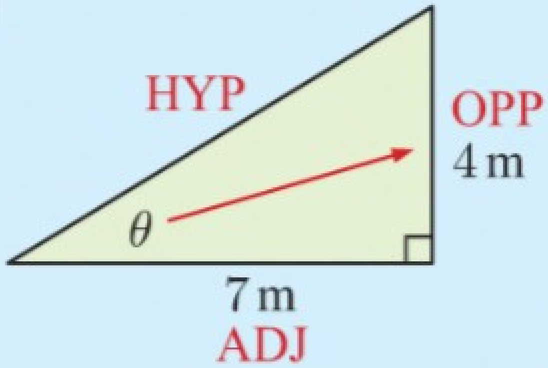
a



b



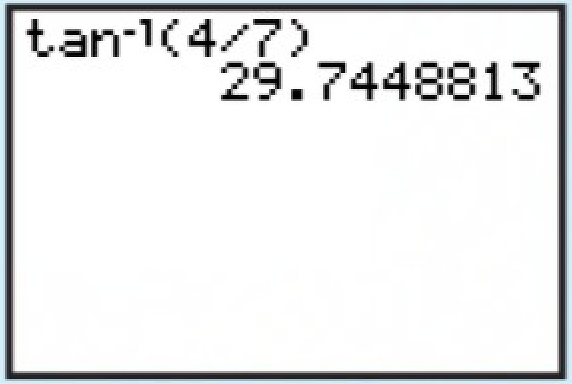
a



$$\tan\theta = \tfrac{4}{7} \qquad \left\{\tan\theta = \frac{\text{OPP}}{\text{ADJ}}\right\}$$

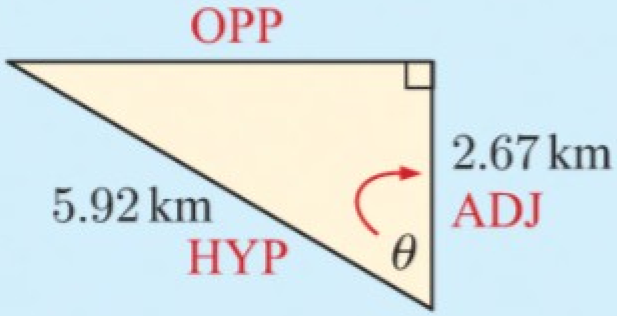
$$\therefore\quad \theta = \tan^{-1}\left(\tfrac{4}{7}\right)$$

$$\therefore\quad \theta \approx 29.7^\circ \qquad \{\text{calculator}\}$$



```
tan⁻¹(4/7)
           29.7448813
```

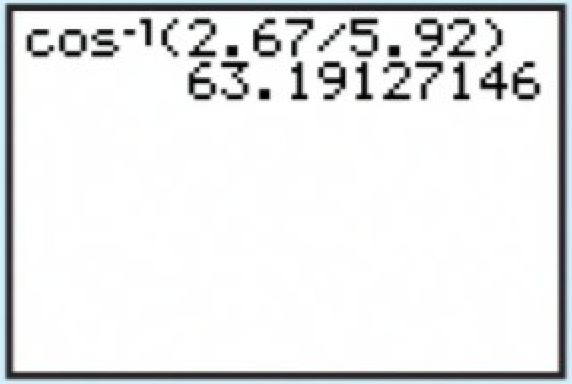
b



$$\cos\theta = \frac{2.67}{5.92} \qquad \left\{\cos\theta = \frac{\text{ADJ}}{\text{HYP}}\right\}$$

$$\therefore\quad \theta = \cos^{-1}\left(\frac{2.67}{5.92}\right)$$

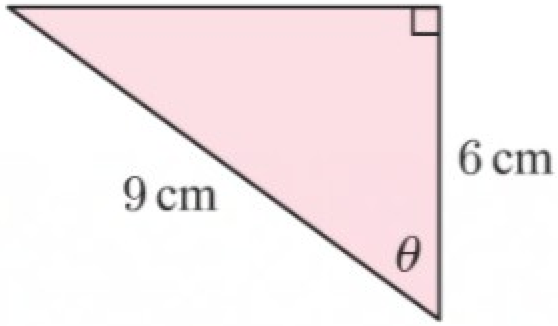
$$\therefore\quad \theta \approx 63.2^\circ \qquad \{\text{calculator}\}$$



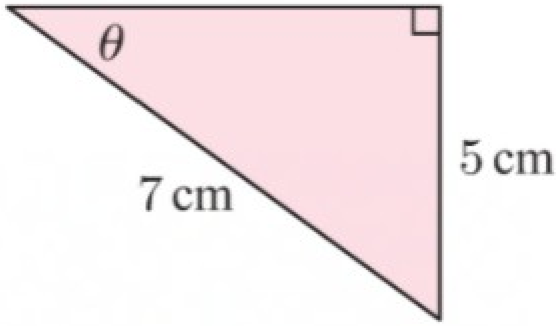
```
cos⁻¹(2.67/5.92)
          63.19127146
```

1 Find the measure of the angle marked θ:

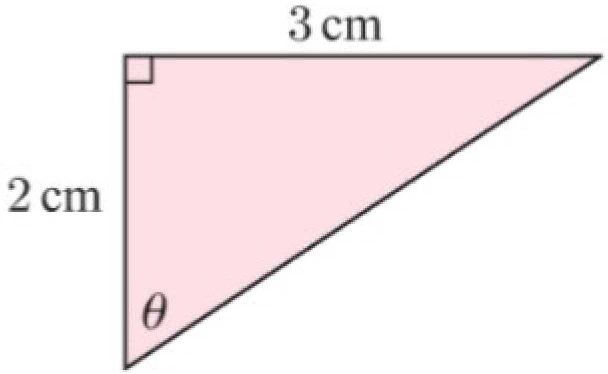
a



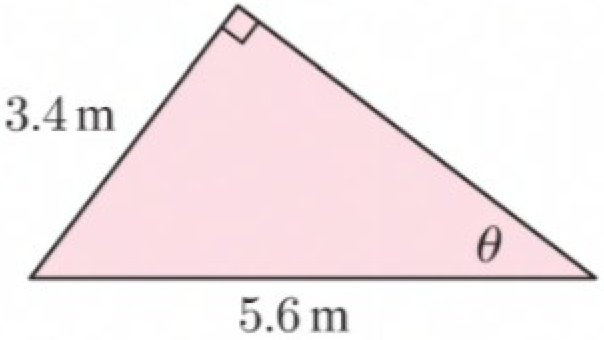
b



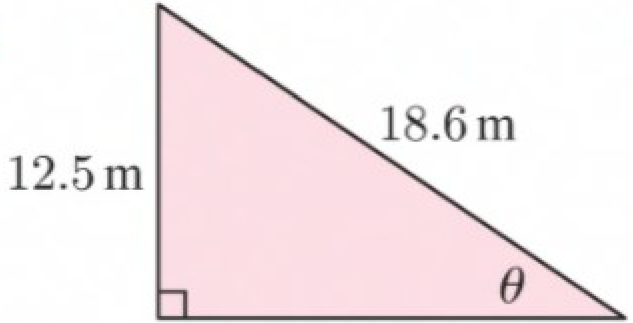
c



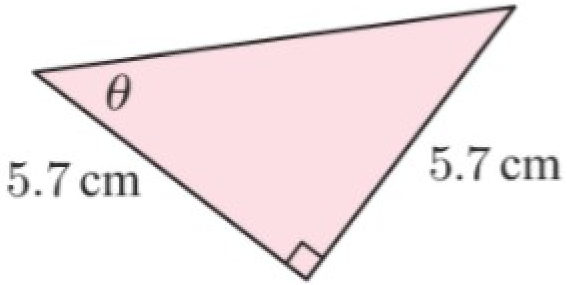
d



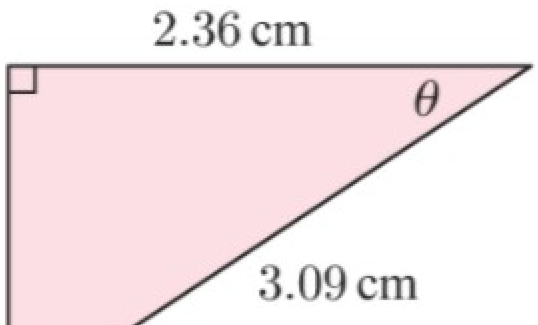
e



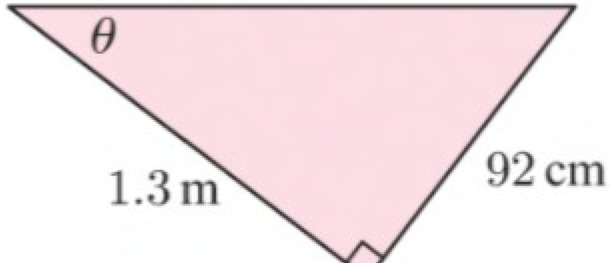
f



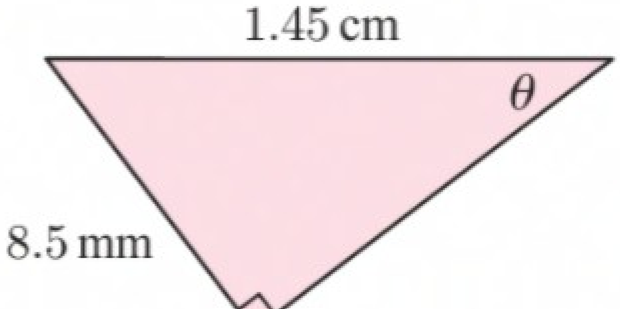
g



h

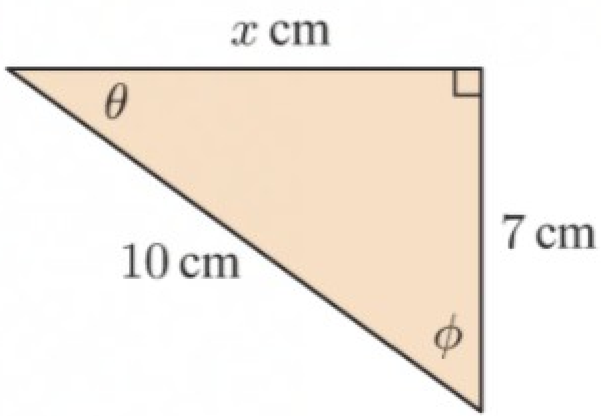


i

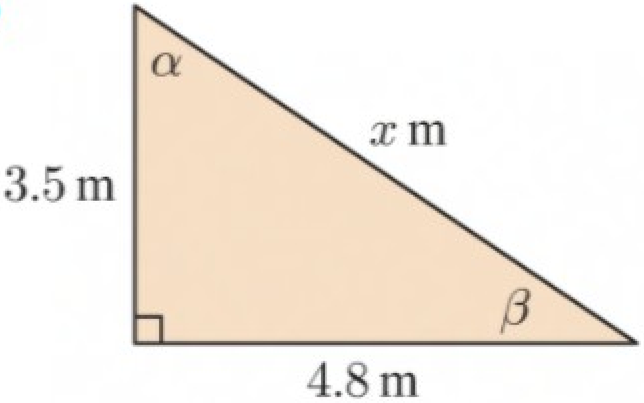


2 Find, using trigonometry, all the unknown sides and angles in the following triangles. Check your answers for x using Pythagoras' theorem.

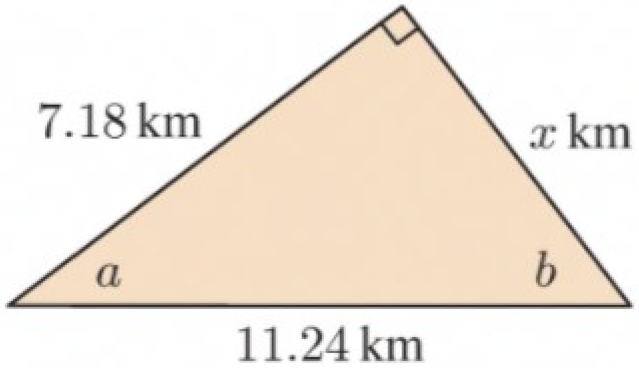
a



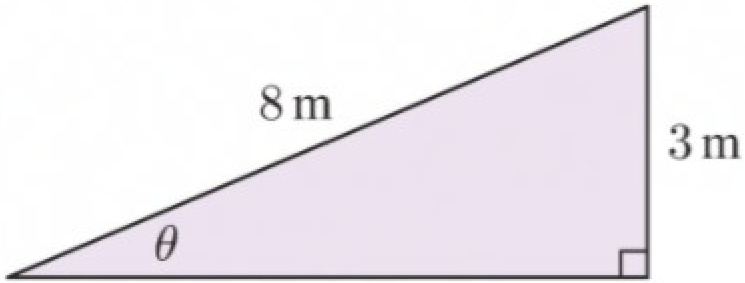
b



c



3 Consider the triangle alongside.



a Copy and complete, stating exact values:

i $\sin\theta = \ldots\ldots$ **ii** $\cos\theta = \ldots\ldots$

iii $\tan\theta = \ldots\ldots$

b Use each of these equations to find θ. Check that you get the same answer each time.

EXERCISE 12A.1

1 **a** $\tan 64^\circ = \dfrac{c}{x}$ **b** $\sin 70^\circ = \dfrac{a}{x}$ **c** $\cos 43^\circ = \dfrac{x}{e}$
d $\sin 35^\circ = \dfrac{x}{b}$ **e** $\cos 61^\circ = \dfrac{g}{x}$ **f** $\tan 30^\circ = \dfrac{h}{x}$

2 **a** $x \approx 12.99$ **b** $x \approx 15.52$ **c** $x \approx 9.84$
d $x \approx 6.73$ **e** $x \approx 3.31$ **f** $x \approx 20.01$
g $x \approx 16.86$ **h** $x \approx 4.45$ **i** $x \approx 1.65$
j $x \approx 26.75$ **k** $x \approx 7.53$ **l** $x \approx 9.61$

3 **a** $\theta = 22^\circ, \quad a \approx 8.09, \quad b \approx 3.03$
b $\theta = 59^\circ, \quad a \approx 13.0, \quad b \approx 7.83$
c $\theta = 65^\circ, \quad a \approx 5.42, \quad b \approx 2.29$

4 ≈ 28.8 cm

EXERCISE 12A.2

1 **a** $\theta \approx 48.2^\circ$ **b** $\theta \approx 45.6^\circ$ **c** $\theta \approx 56.3^\circ$

d $\theta \approx 37.4^\circ$ **e** $\theta \approx 42.2^\circ$ **f** $\theta = 45^\circ$

g $\theta \approx 40.2^\circ$ **h** $\theta \approx 35.3^\circ$ **i** $\theta \approx 35.9^\circ$

2 **a** $\phi \approx 45.6^\circ, \quad \theta \approx 44.4^\circ, \quad x \approx 7.14$

b $\alpha \approx 53.9^\circ, \quad \beta \approx 36.1^\circ, \quad x \approx 5.94$

c $a \approx 50.3^\circ, \quad b \approx 39.7^\circ, \quad x \approx 8.65$

3 **a** **i** $\sin\theta = \frac{3}{8}$ **ii** $\cos\theta = \frac{\sqrt{55}}{8}$ **iii** $\tan\theta = \frac{3}{\sqrt{55}}$

b In each case, $\theta \approx 22.0^\circ$.